Diffusion coefficient of piecewise linear maps

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By using the cycle expansion, we obtain general expressions for the determination of the diffusion coefficient D of a piecewise linear map which is parametrized by k and $\tilde{h}$ (where the map contains $2k+5$ branches of line segment, and $\tilde{h}$ is the height of the shortest line). By restricting $\tilde{h}=\beta/m$ [$\beta=1,\ldots,(k+1)/2$; m is the slope of the map], a closed form expression of D can be obtained and some of its consequences are discussed. The limiting form of D ($k\to\infty$) is then shown to be k^2. For the simplest case with $k=1$, we also show that more exact results can be found. A limiting case with $\tilde{h}\to 0$ is discussed where agreement with the result obtained from the invariant measure approach is established.

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I. INTRODUCTION

Diffusion is an old problem which has been discussed in physics and mathematics since the end of the last century. One of the famous mechanisms is given by the thermal agitation of the molecules and such motion has been named Brownian motion. Theoretically, Brownian motion can be modeled by particle motion under the influence of a stochastic force term.

In recent years, diffusion has been realized in dynamical systems in which the dynamics is completely specified and it is called deterministic diffusion. In this kind of diffusion, the mechanism is due to the chaotic motion of the system. This is an interesting phenomenon in which stochastic behavior is generated by a deterministic system. It is also interesting to note that deterministic diffusion can be related to a real physical system such as Josephson junctions driven by microwave radiation [1]. Recently, some exact results of the diffusion coefficient D has been evaluated by Artuso [2]. By employing the cycle expansion techniques developed by Artuso, Aurell, and Cvitanovic [3], Artuso was able to obtain expressions which led to the exact determination of D in a whole class of models. However, in [2], the symbolic dynamics are unrestricted. It is of great interest to see if the same method can also be extended to cases where pruning can occur. In a recent paper, Tseng *et al.* [4] has indicated that in some special cases of the simplest mapping with pruning, D can also be obtained. In this work, based on the approach of [3] we provide a general discussion of the exact results of the general piecewise linear map where pruning is allowed. It is interesting to note that for all cases a general classification of the pruning rule can be established and hence exact expressions are obtained within the context of cycle expansion. As a consequence, closed form expression of D is obtained in a subclass of these models, and also the large k limit of D can be found. Furthermore, we have found a new class of exact results in the simplest model. By exploring the limit of these results, we are able to show a scaling limit exists which is in agreement with the invariant measure approach.

This paper is organized in the following manner. In Sec. II, we briefly review the method of cycle expansion. Then we discuss the general piecewise linear map in Sec. III, all exact expressions for determining D are given in this section. The closed form expression of D is given for a subclass of maps and some of its consequences are discussed in Sec. IV. More exact results of the simplest model are treated in Sec. V, where a scaling limit is established and also shown to be in agreement with previous work. In the final section a brief discussion on some general results is given. The Appendix contains all the pruning rules and symbolic dynamics of the main text.

II. CYCLE EXPANSION

In this section, the method of cycle expansion is briefly reviewed; more details and notations can be found in [3]. Cycle expansion is a perturbation theory for chaotic systems of low dimensional phase space. The essence of this method is to express averages over chaotic orbits in terms of unstable short periodic orbits. The contribution of long periodic orbits is expected to be exponentially small and hence a meaningful expansion can be obtained. The mechanism for small contribution from long periodic orbits is due to cancellation. For simple cases such as piecewise linear mapping (which are treated in this work) the cancellation is exact and the results are then completely given in terms of a few short periodic orbits.

Cycle expansion is an expansion on the dynamical ζ function of a dynamical system. The ζ function can be obtained by the transfer operator technique. The transfer operator T is a linear evolution operator of the system (such as the one used in the evaluation of the escape rate of a repeller [4]) which determines the evolution of the system under the deterministic map $x_{n+1}=f(x_n)$. For example, the kernel of the escape rate of a repeller is

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$$T(y,x)=\delta(y-f(x)) . \quad (1)$$

Since the evolution of the system is completely determined by T, the evaluation of its eigenspectrum is the most important issue in the discussion of its dynamical properties. It is easy to see that the eigenspectrum of T is related to the following determinant:

$$\det(1-zT)=\exp[\mathrm{Tr}\ln(1-zT)] = \exp\left[-\sum_{n=1}^{\infty}\frac{z^n}{n!}\mathrm{Tr}(T^n)\right] . \quad (2)$$

Due to the fact that $\mathrm{Tr}(T^n)$ picks up contributions from all repeats of prime cycles p

$$\mathrm{Tr}(T^n)=\sum_{n_p|n} n_p t_p^{n/n_p} , \quad (3)$$

where $n_p|n$ denotes that n_p is a divisor of n, the above determinant can be rewritten as

$$\det(1-zT)=\prod_p (1-z^{n_p}t_p) . \quad (4)$$

In (4) all prime cycles p should appear in the product. The dynamical ζ function is then defined as

$$\zeta_0^{-1}(z)=\det(1-zT)=\prod_p (1-t_p) . \quad (5)$$

Here we have absorbed the z factors into t_p: $z^{n_p}t_p \to t_p$. We would like to emphasize that Eq. (5) is exact and no approximation has been used up to this point. It is now clear that the eigenspectrum of T is the zeros of ζ_0^{-1}.

To illustrate how cycle expansion can be done, we expand the Euler product (5)

$$\zeta_0^{-1}=\prod_p (1-t_p)=1-\sum_{p_1p_2\cdots p_k} t_{p_1+p_2+\cdots+p_k} , \quad t_{p_1+p_2+\cdots+p_k}=(-1)^{k+1}t_{p_1}t_{p_2}\cdots t_{p_k} . \quad (6)$$

Since the expansion contains z to all orders, it is necessary to assume z being small enough such that the infinite sum makes sense. Therefore all manipulations are only done formally. Fortunately, due to cancellation, the infinite sum is truncated to a finite sum and the requirement of having small z goes away. The next step is to reorganize the various terms in (6) in a definite manner which also provides a way to see how cancellation can occur. To make the discussion more concrete, we will take the binary dynamics as an example. For this case the above expansion can be written explicitly

$$\zeta_0^{-1}=(1-t_0)(1-t_1)(1-t_{10})(1-t_{100})\cdots = 1-t_0-t_1-t_{10}-t_{100}-t_{110}\cdots-t_{0+1} -t_{0+01}-\cdots , \quad (7)$$

where the prime cycles are denoted by the symbolic sequences of two unrestricted symbols $\{0,1\}$. The reorganization is done by grouping the terms of the same total symbol string length

$$\zeta_0^{-1}=1-t_0-t_1-[t_{01}-t_0t_1] -[(t_{100}-t_{10}t_0)+(t_{101}-t_{10}t_1)]-\cdots . \quad (8)$$

It is obvious in this expansion that t_0 and t_1 are the most important quantities since all longer orbits can be pieced together from them approximately. Therefore all the periodic orbits which cannot be approximated by shorter orbits are called fundamental cycles. In the above example, the fundamental cycles are t_0 and t_1. The terms of the same total length which are grouped together in the brackets of (8) are called the curvature corrections c_n, where n denotes the total length. If all curvature corrections vanish, then ζ_0^{-1} is exactly given in terms of the fundamental cycles and this is the spirit of cycle expansion. For the case in which c_n are nonvanishing, cycle expansion provides a systematic way to carry out corrections. For the case of binary dynamics generated by the tent map, it can be shown that all curvature corrections vanish identically. This is due to the uniform hyperbolicity in this particular map. In this case, the ζ_0^{-1} is a linear equation in z and its zero can be computed explicitly. For other cases, ζ_0^{-1} is given by a polynomial of z of finite degree. The finite degree of ζ_0^{-1} arises from two possibilities. Either (a) due to vanishing curvature corrections or (b) by truncation. The last possibility can only be justified by the convergence of the problem to the next leading order.

III. PIECEWISE LINEAR MAP AND DIFFUSION COEFFICIENT

The piecewise linear map is defined as follows:

$$x_{n+1}=f(x_n) , \quad f(x+n)=f(x)+n , \quad (9)$$

and the map f (see Fig. 1) is defined on the interval

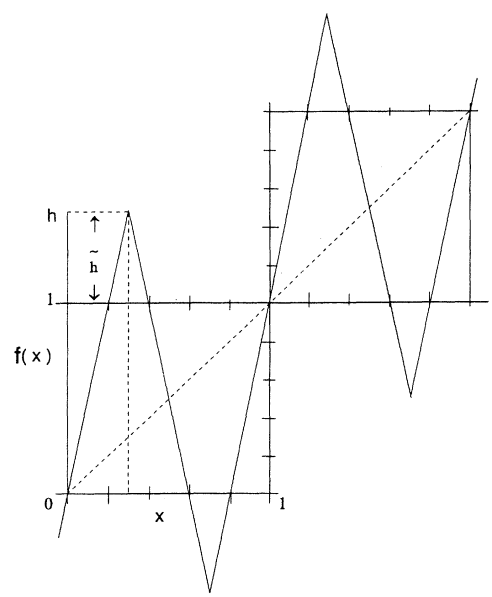

FIG. 1. The graph of $f(x)$, in this graph $h=1+\tilde{h}$.

$I=[0,1]$:

$$f(x)=\begin{cases} mx, & x\in\left[0,\dfrac{h}{m}\right] \\ -mx+2h, & x\in\left[\dfrac{h}{m},\dfrac{3h-1}{m}\right] \\ mx+(1-m), & x\in\left[\dfrac{3h-1}{m},1\right] \end{cases}, \tag{10}$$

where h is the absolute maximum of f on I. An associate map $\tilde{f}$ can be constructed from f by defining $\tilde{f}=f$ mod 1. As pointed out by Artuso [2], the diffusion coefficient D of f can be obtained by considering an appropriate ζ function defined by the associate map $\tilde{f}$. For this problem, the ζ function is

$$\zeta_0^{-1}(z,\alpha)=\prod_p [1-z^{n_p}\exp(\sigma_p\alpha)/|\Lambda_p|] , \tag{11}$$

where p denotes the prime cycles of $\tilde{f}$ and n_p being the period of prime cycle p; σ_p is the integer part of $\tilde{f}^{n_p}$ at one of periodic point x_p and

$$\Lambda_p=\left.\frac{d\tilde{f}^{n_p}}{dx}\right|_{x=x_p} . \tag{12}$$

For the linear map considered in this work, $|\Lambda_p|=m$. The diffusion coefficient D is then determined by

$$D=-\tfrac{1}{2}\left.\frac{\partial^2 z_c}{\partial\alpha^2}\right|_{\alpha=0} , \tag{13}$$

where z_c is the smallest zero of ζ_0^{-1} defined in (11).

In general, the graph of $\tilde{f}$ consists of $2k+5$ line segments (Fig. 2), where k is an odd integer. The line segments can be divided into two groups which have slopes m and $-m$, respectively. For the lines with slope m, the number of line segments is $k+3$ and the remaining $k+2$ line segments have slope $-m$. For the case with h being an integer, the range of $\tilde{f}$ of these lines covers the interval [0,1]. However, the h being a noninteger, there are two line segments with the maximum value of $\tilde{f}$ less than unity and they are labeled by a and b; furthermore, there are another two line segments with the minimum value of $\tilde{f}$ greater than zero and they are labeled by c and d. The maximum value of $\tilde{f}$ for a and b is parametrized by $\tilde{h}$. The rest of the lines have the same maximum value of $\tilde{f}$ which is equal to 1 and they are labeled by $1,2,\ldots,2k+1$ (Fig. 2). Thus, the associate map $\tilde{f}$ is characterized by two parameters k and $\tilde{h}$. In the following, we shall establish the fact that for each case of fixed k, there are $2k+5$ different values of $\tilde{h}$ which lead to exact expressions for determining D. Moreover, it is interesting to note that, for all cases, the expressions are at most cubic polynomials in z.

These $2k+5$ different values of $\tilde{h}$ are determined by the following relations:

$$m=4\tilde{h}+2k+1 , \tag{14a}$$

$$m=\frac{\tilde{h}}{\lambda} , \tag{14b}$$

and

$$\tilde{h}=\begin{cases} \dfrac{\beta}{m} \\ \dfrac{K_+}{m}+l\lambda \\ \dfrac{K_++\beta'}{m}+2\lambda \\ \dfrac{K_++k}{m}+(l+1)\lambda \\ \dfrac{K_++k+\beta}{m}+4\lambda, \end{cases} \tag{14c}$$

where $\beta=1,\ldots,K_+$; $l=1,2$; $\beta'=1,\ldots,k$ and λ is the width of the projection of line segment a on the x axis. Here we define $K_\pm=(k\pm1)/2$ where K_- will be needed in later discussion. It is obvious that $\tilde{h}$ can be classified into six classes which are just corresponding to six different kinds of symbolic dynamics. For the case with $\tilde{h}=1$, this is corresponding to $\lambda=1/m$, $\beta=K_+$, and $m=2k+5$. This case has been already discussed in [2]. For other values of $\tilde{h}$ all discussions on the pruning rules and the corresponding symbolic dynamics of these cases are given in the Appendix. Here, we only quote the expressions of the cycle expansion and the expressions which determine z_c. In what follows, the slope m can be determined from Eq. (14).

We proceed in the order of increasing $\tilde{h}$, the first class is $h=\beta/m$ where $\beta=1,\ldots,K_+$. For this class, the symbolic dynamics is simple and the ζ function is

$$\zeta_0^{-1}(z,\alpha)=1-\sum_{i\in Q}t_\alpha-\sum_{\alpha\in A}\sum_i{}' t_{\alpha i} , \tag{15}$$

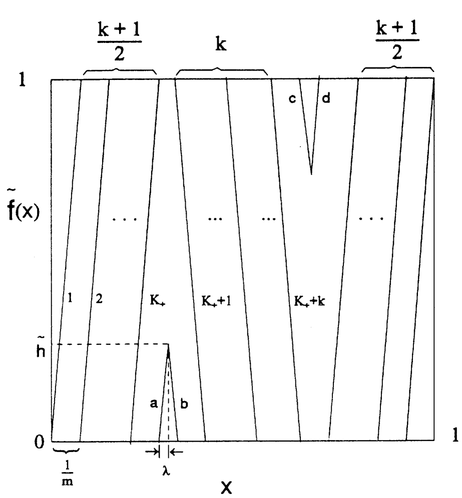

FIG. 2. The graph of the associated map $\tilde{f}(x)$. The labels of some line segments are given.

where $A=\{a,b,c,d\}$; $Q=\{1,\ldots,\beta;2k+1-\beta,\ldots,2k+1\}$, and the restricted sum $\sum_i'$ restricts $i=1,\ldots,\beta$ for $\alpha=a$ or b and $i=2k+1-\beta,\ldots,2k+1$ for $\alpha=c$ or d. The equation to determine the zero of ζ_0^{-1} is

$$1-\frac{z}{m}\sum_{n=0}^{K-}a_n\cosh n\alpha-\frac{4z^2}{m^2}\sum_{n=0}^{\beta-1}\cosh(K_++n)\alpha=0\ . \tag{16}$$

Here, $a_0=3$ and $a_n=4$ for $n\neq0$. Since this is a quadratic equation in z, the solution of z can be obtained analytically. We will come back to this point in the next section. The evaluation of D is straightforward and will also be discussed in Sec. IV.

The next case is corresponding to $\tilde{h}=(K_+/m)+\lambda$. This case has a fixed point, its symbolic dynamics becomes interesting and will be discussed in the Appendix. The ζ function is

$$\zeta_0^{-1}(z,\alpha)=1-t_a-t_d-\sum_{i\in Q}t_i-\sum_{\gamma\in\Gamma}t_{b\gamma}-\sum_{\bar\gamma\in\bar\Gamma}t_{c\bar\gamma}+t_at_d+t_a\left[\sum_{i\in(\bar\Gamma\cup\bar Q)}t_i+\sum_{\bar\gamma\in\bar\Gamma}t_{c\bar\gamma}\right]+t_d\left[\sum_{i\in(\Gamma\cup\bar Q)}t_i+\sum_{\gamma\in\Gamma}t_{b\gamma}\right]-t_at_d\sum_{i\in\bar Q}t_i\ , \tag{17}$$

where $Q=\{1,\ldots,K_+;(3k+1)/2,\ldots,2k+1\}$; $\Gamma=\{1,2,\ldots,K_+\}$, and $\Gamma=\{(3k+1)/2,\ldots,2k+1\}$; $\bar Q=\{K_++1,\ldots,K_-+k\}$. The corresponding equation for z is

$$1-\frac{z}{m}\sum_{n=0}^{K_+}a_n\cosh n\alpha+\frac{z^2}{m^2}+\frac{2z^3}{m^3}\sum_{n=0}^{K_-}b_n\cosh(K_+-n)\alpha=0\ , \tag{18}$$

where $a_{K_+}=2$ and for $n\neq K_+$, a_n are the same as the previous case. Here, $b_0=1$ and $b_n=2$ for $n\neq0$. Since (18) is a cubic equation, its analytic solution can also be obtained and hence exact evaluation of D can be performed.

The third case has new structure in the symbolic dynamics. For $\tilde{h}=[(K_++\beta)/m]+2\lambda$, with $\beta=0,1,\ldots,K_-$. The ζ function is defined by

$$\zeta_0^{-1}(z,\alpha)=\prod_{p\in\Phi}(1-t_p)\prod_{p'\in\Phi'}(1-t_{p'})\ , \tag{19}$$

where Φ is a set in which unrestricted symbolic dynamics can be defined and Φ' denotes the set of all allowed prime cycles excluded by Φ. Here we have

$$\Phi\{1,\ldots,2k+1,s|s\in T_\alpha,\alpha=a,b,c,d,\}\ . \tag{20}$$

The symbols T_α are defined in the Appendix. For Φ' we have

$$\Phi'=\{p|p\in T'_\alpha,\alpha=a,b,c,d\}\ . \tag{21}$$

Again T'_α can be found in the Appendix. By expanding out (19), the equation for z is

$$1-\frac{z}{m}\sum_{n=0}^{K_+}a_n\cosh n\alpha+\frac{4z^2}{m^2}\sum_{n=0}^{2K_--\beta}b_n\cosh(k-\beta-n)\alpha-\frac{4z^3}{m^3}\sum_{n=0}^{K_--\beta}d_n\cosh n\alpha=0\ . \tag{22}$$

All a_n are defined as before, and the coefficients b_n are

$$b_n=\begin{cases}1, & n<K_--\beta\\ 2, & n\geq K_--\beta.\end{cases} \tag{23}$$

For d_n, we have $d_0=1$ and $d_n=2$ for $n\neq0$. It is noted that Eq. (22) is again a cubic equation. This remains to be true in the next couple of cases. However, as $\tilde{h}>[(2k+1-K_+)/m]+3\lambda$, then the degree of the equation becomes quadratic again as will be shown later.

For $\tilde{h}=[(k+1+\beta)/m]+2\lambda$, with $\beta=0,1,2,\ldots,K_-$, the symbolic dynamics are the same (see the Appendix). Hence we obtain

$$1-\frac{z}{m}\sum_{n=0}^{K_+}a_n\cosh n\alpha+\frac{4z^2}{m^2}\sum_{n=0}^{K_-}b_n\cosh(K_+-n)\alpha+\frac{4z^3}{m^3}\sum_{n=0}^{\beta}d_n\cosh n\alpha=0\ . \tag{24}$$

Here, a_n and d_n are defined as before; the coefficients b_n are

$$b_n=\begin{cases}1, & n\leq\beta\\ 2 & \text{otherwise.}\end{cases} \tag{25}$$

As $\tilde{h}$ reaches the value of $[(3k+1)/2m]+3\lambda$, the T_α and T'_α defined earlier have a new feature. This is due to the fact that mixing among $\{a,b,c,d\}$ arises. The contributions from Φ' is

$$\prod_{p'\in\Phi'}(1-t_{p'})=(1-t_a)(1-t_d)\left[1-\sum_{k=0}^{\infty}t_{cd^k}-\sum_{k=0}^{\infty}t_{ba^k}\right]. \tag{26}$$

This expression can be simplified by expanding out the right-hand side of (26), and we have

$$\prod_{p\in\Phi'}(1-t_{p'})=1-\sum_{\alpha\in A}t_\alpha+t_at_c+t_bt_d+t_at_d\ . \tag{27}$$

By putting Eq. (27) and the contribution from Φ into Eq.

(19), and also due to uniform hyperbolicity we have the equation of z as

$$1-\frac{z}{m}\sum_{n=0}^{K_+} a_n \cosh n\alpha+\frac{3z^2}{m^2}+\frac{4z^2}{m^2}\sum_{n=0}^{K_-}\cosh(K_+-n)\alpha - \frac{z^3}{m^3}=0 \ , \quad (28)$$

where a_n is defined in the previous discussions.

The last case in which the coefficient D can also be obtained exactly is when $\tilde{h}=[(3k+1+\beta)/m]+4\lambda$, where $\beta=0,1,2,\ldots,K_+$. As pointed out earlier, the cases with $\beta=K_+$ have been discussed by Artuso [2] and we include them here for completeness. The cycle expansion of the ζ function is still given by the product of two factors, namely, the contributions from the set Φ and Φ', respectively. From Φ and Φ' given in the Appendix we can routinely obtain the equation

$$1-\frac{z}{m}\sum_{n=0}^{K_+} a_n \cosh n\alpha+\frac{4z^2}{m}\sum_{n=0}^{K_--\beta}\cosh(K_+-n)\alpha=0 \ , \quad (29)$$

where a_n is defined as usual. The interesting point about this case is that it is a quadratic equation and a closed form solution of z can be evaluated easily.

In passing, we observe that the results obtained in [2,3] are just special cases in our discussions. In [2], Artuso has discussed the situation where $\tilde{h}=1$ and k is an arbitrary integer. Whereas in [4], Tseng *et al.* have considered the case where $k=1$ with $\tilde{h}$ being the allowed values of (14). Furthermore, our results indicate clearly that the equation for determining z is at most a cubic equation of z. However, we should stress that this is only true for the cases at hand. For other values of $\tilde{h}$, higher power of z can appear. This can be seen in Sec. IV where explicit examples are given. The reason for having an equation with a degree less than four can be traced back to the fact that the fundamental cycles only appear in the form of t_i and t_{ij} which contain at most z^2. The product of contributions from Φ and Φ' then leads to an equation which is at most cubic in z.

We would like to emphasize that all these results are obtained within the framework of cycle expansion. It indicates the effectiveness of the method in analytical calculation in chaotic system. In the next section, we shall evaluate the closed form expression of D for $\tilde{h}=\beta/m$ ($\beta=1,2,\ldots,K_+$). This cannot be done without using the cycle expansion.

IV. CLOSED FORM EXPRESSION OF D

From the previous discussions, we observe that there are certain cases in which the zeros of the ζ function are determined by a quadratic equation. Let us now concentrate on the case where $\tilde{h}=\beta/m$ ($\beta=1,\ldots,K_+$). Since the diffusion coefficient D is evaluated at $\alpha=0$, it is more convenient to work with a polynomial in α. Furthermore, we only have to expand everything up to the order of α^2, terms of higher order of α do not contribute to D. In fact, the solution of z as a function of α can be solved perturbatively with α being a small parameter. It turns out that such an approximation scheme provides a nice expression of D. The zeroth order solution z_0 can be obtained by setting $\alpha=0$ in ζ_0^{-1}. From Eq. (16) we have

$$1-(3+4K_-)\frac{z_0}{m}-4\beta\frac{z_0^2}{m^2}=0 \ . \quad (30)$$

By using the fact that $K_-=(k-1)/2$, we have

$$1-(2k+1)\frac{z_0}{m}-4\beta\frac{z_0^2}{m^2}=0 \ . \quad (31)$$

It is interesting to note that the relevant solution z_0 of (31) is exactly equal to 1. This is due to the fact that the equation determining the slope m [from Eq. (14)] is

$$m^2-(2k+1)m-4\beta=0 \ , \quad (32)$$

which is exactly Eq. (31) with $z_0=1$. Hence the perturbative solution of z_c is given by

$$z_c=z_0+\delta=1+\delta \ . \quad (33)$$

Expanding (16) in powers of α and using $z_c=1+\delta$, we have

$$\delta=-\frac{2\alpha^2 m}{m^2+4\beta}\left\{\frac{k}{6}K_+K_-+\frac{\beta K_+^2}{m}+\frac{\beta}{m}(\beta-1)[K_++\tfrac{1}{6}(2\beta-1)]\right\} . \quad (34)$$

It is obvious that the absolute value of the coefficient of α^2 in (34) is just the diffusion coefficient

$$D=\frac{2m}{m^2+4\beta}\left\{\frac{k}{6}K_+K_-+\frac{\beta K_+^2}{m}+\frac{\beta}{m}(\beta-1)[K_++\tfrac{1}{6}(2\beta-1)]\right\} . \quad (35)$$

We would like to mention that for D to be a diffusion coefficient, it has to be positive. This fact can be seen explicitly in (35) where each term is positive definite. Thus, the minus sign in Eq. (34) is crucial.

By having the closed form of D, the large k behavior of D can be obtained. For the case where $\beta \ll k$, Eq. (35) can be approximated by the following expression:

$$\lim_{k\to\infty} D=\frac{k^2}{24}-\frac{k}{12}\beta \quad (\beta \ll k) \ . \quad (36)$$

Therefore, the large k limit of D in this case ($\tilde{h}=\beta/m$) is dominated by the k^2 term and as a result D diverges as $k\to\infty$. In the same way, we can also evaluate D in the large k limit for $\tilde{h}=(3k+\beta+1)/m$. In this case it can be shown that D also approaches infinity in the form of k^2. It is not hard to see how such scaling of D appears. In this limit, there are some general properties that can be seen from the equations which determine the slope m of the model and the solution z_0 of the zeroth order approximation. First of all, the equation of m is a quadratic equation and it is obvious that m is proportional to k as

$k \to \infty$. Secondly, from the zeroth order equation of z, it is easy to see that z_0 is of order of unity as $k \to \infty$. Thus, z_0 is insensitive to k in the large k limit. Therefore, the diffusion coefficient D can then be given by the correction of z_0, for example, the δ term in Eq. (33). However, the largest term in δ corrections is the term of the form $(1/m)\sum n^2$ where the upper limit of the summation is of order k. Since $\sum n^2$ is of order k^3 and m scales as k, thus we have $D \sim k^2$ and D diverges as $k \to \infty$.

It is natural to question the large k limit of D of the other cases in which z is determined by a cubic equation. Since the slope m still scales as k, z_0 is of the order of 1 which is insensitive to k as $k \to \infty$. Moreover, the cubic equation is also simplified in the large k limit. This is again due to the fact that $z_0 \sim 1$ and $m \sim k$. Effectively the large k requirement reduces the cubic equation to a quadratic equation. Therefore the correction of z, denoted again by δ, is also dominated by the term of the type $\sum n^2$. As a result, the scaling limit of D in all cases should be k^2 as k approaches infinity. It remains to be seen whether $D \sim k^2$ is true for all other cases (with $\tilde{h}$ being arbitrary) as $k \to \infty$.

V. SCALING LIMIT OF D WITH $k=1$

In this section, we return to the case where $k=1$; $\tilde{h} < 1/m$ and $m < 4$. In fact, there exists an infinite number of cases in which D can be evaluated exactly. Furthermore, a limiting case $\tilde{h} \to 0$ will be discussed and it agrees with previous study.

These new $\tilde{h}$'s are generated by the following relations:

$$\tilde{h} = \tfrac{1}{4}(m-3) \tag{37}$$

and

$$m^l \tilde{h} = 1, \quad l = 1,2,3,\ldots \,. \tag{38}$$

These equations imply

$$m^{l+1} - 3m^l - 4 = 0 \tag{39}$$

where $l=1$ is the case (with $m=4$ and $\tilde{h}=\frac{1}{4}$) discussed in [3] and also included in our previous discussions.

It is easy to obtain the symbolic dynamics of these cases since no pruning arises here. The symbolic dynamics which is complete and is generated by the following symbols

$$\Phi = \{1,2,3,aI_l,bI_l,cI_l',dI_l'\} \,, \tag{40}$$

where I_l is the shorthand notation for a series of 1 of length l, namely,

$$aI_l = a1\cdots 1 \,. \tag{41}$$

Similarly, I_l' is the shorthand notation for a series of 3 of length l.

The cycle expansion of the ζ function is given by the shortest prime periods

$$\zeta_0^{-1} = 1 - t_1 - t_2 - t_3 - t_{aI_l} - t_{bI_l} - t_{cI_l'} - t_{dI_l'} \,, \tag{42}$$

and the equation for z is

$$1 - 3\frac{z}{m} - 4\cosh\alpha \frac{z^{l+1}}{m^{l+1}} = 0 \,. \tag{43}$$

By using the same technique, we have the equation of z_0 (by setting α to zero)

$$1 - 3\frac{z_0}{m} - 4\frac{z_0^{l+1}}{m^{l+1}} = 0 \,. \tag{44}$$

It is obvious that $z_0 = 1$ is a solution since Eq. (39) is the same as Eq. (44) with z_0 replaced by 1. Thus the diffusion coefficient is

$$D = \frac{2}{3m^l + 4(l+1)} \quad (3 \le m \le 4;\ l = 1,2,\ldots) \,, \tag{45}$$

where m is given by (39). The case with $m=3$ is corresponding to $l \to \infty$. Thus it is interesting to see how D varies in the large l limit. In this limit, the height $\tilde{h}$ approaches 0.

The approximation of m can be written as $m = 3+\epsilon$, and $\tilde{h} = \epsilon/4$. Within this approximation, $\epsilon = 4/3^l$, and D is approximated by

$$D \simeq \tfrac{1}{6}\epsilon = 2\lambda \,, \tag{46}$$

where $\lambda = \epsilon/12$ has been used. Therefore D is linearly proportional to the height $\tilde{h}$ and approaches zero as $\tilde{h} \to 0$. This result has been obtained by Grossmann and Fujisaka [5] by using the invariant measure approach.

VI. CONCLUSIONS

In this work, we have shown that for piecewise linear maps, deterministic diffusion can be discussed through periodic orbits. This is being done by introducing the appropriate symbolic dynamics and as a result all prime periods of the system are identified. Employing the cycle expansion of these maps, we have shown that all curvature terms c_n vanish and hence a convergent expansion is reduced to an exact finite polynomial. Therefore exact determination on the diffusion coefficient D becomes feasible. The effectiveness of the cycle expansion is also clearly shown in our results, as closed form solutions can be obtained for a whole class of models. From our results, we conclude that the diffusion coefficient D is proportional to k^2 in the large k limit. It is conjectured that such a scaling form is a general property of piecewise linear map as k is large. Furthermore, by realizing the fact that D is evaluated at $\alpha=0$, an expansion in α is employed in this work and interesting results such as positiveness of D can be shown explicitly.

We have also found a whole sequence of $\tilde{h}$ with $\tilde{h} \le 1/m$ $(m \le 4)$. This case and its generalization deserve further studies. The generalization can be considered in two directions. One obvious way to generate more solvable models of $\tilde{h}$ is to extend our discussion in Sec. V $(0 \le \tilde{h} \le 1/m)$ to any two consecutive values of $\tilde{h}$ that we have discussed in Sec. III. We expect that infinite numbers of new $\tilde{h}$ can be found in this way. The other possibility is to generate more $\tilde{h}$ in between any two consecutive values of $\tilde{h}$ which are determined in Sec. V. It is obvious that a hierarchy structure appears and we expect that some sort of self similarity structure of D

might exist. This line of work is now under investigation and will be reported elsewhere.

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APPENDIX: THE PRUNING RULES AND THE SYMBOLIC DYNAMICS

In this appendix, we collect all the results of the pruning rules and the corresponding symbolic dynamics of the cases discussed in the main text.

(a) $\tilde{h}=\beta/m$ $(\beta=1,\ldots,K_+)$. The forbidden sequences are as follows: *-al-*, *-aa-*, *-bl-*, *-bb-*, *-cl′-*, *-cc-*, *-cd-*, *-dl′-*, *-dd-*, where $l=\beta+1,\beta+2,\ldots,2k+1$, and $l'=1,\ldots,2k-\beta$. It follows that a and b must always be followed by i whereas c and d are always followed by i'. Here, $i\in\{1,2,\ldots,\beta\}$ and $i'\in\{2k+1-\beta,2k+2-\beta,\ldots,2k+1\}$. The symbolic dynamics is thus given by the following unrestricted symbols: $\{1,2,\ldots,2k+1;ai,bi,ci',di'\}$.

(b) $\tilde{h}=(K_+/m)+\lambda$. The forbidden sequences are as follows: *-al-*, *-bl-*, *-cl′-*, *-dl′-*, *-ab-*, *-ac-*, *-ad-*, *-bb-*, *-bc-*, *-bd-*, *-ca-*, *-cb-*, *-cc-*, *-da-*, *-db-*, *-dc-*, where l and l' are defined as in the previous case. For this case, a and d have fixed points and they are denoted by $\bar{a}$ and $\bar{d}$, respectively. These fixed points should be included in the cycle expansion, since they are not pruned. The unrestricted symbols are as follows: $\{\bar{a},\bar{d};i,a^k\gamma,ba^k\gamma,d^k\bar{\gamma},cd^d\bar{\gamma}\}$, where $k=0,1,2,\ldots$ and $i\in\bar{Q}=\{K_++1,\ldots,K_-+k\}$; $\gamma\in\Gamma=\{1,2,\ldots,K_+\}$ and $\bar{\gamma}\in\bar{\Gamma}=\{(3k+1)/2,(3k+3)/2,\ldots,2k+1\}$.

(c) $\tilde{h}=[(K_++\beta)/m]+2\lambda$ $(\beta=0,1,\ldots,K_-)$. Here, the forbidden sequences are as follows: *-al-*, *-bl-*, *-cl′-*, *-dl′-ac-*, *-ad-*, *-bc-*, *-bd-*, *-ca-*, *-cb-*, *-da-*, *-db-*. $l'=1,2,\ldots,[(3k-1)/2]-\beta$. The unrestricted grammar is given by the set Φ:

$$\Phi=\{1,\ldots,2k+1,s|s\in T_\alpha,\ \ \alpha=a,b,c,d\}\ , \qquad \text{(A1)}$$

where T_α is the set of all dotted elements in tree $\hat{T}_\alpha$ (see

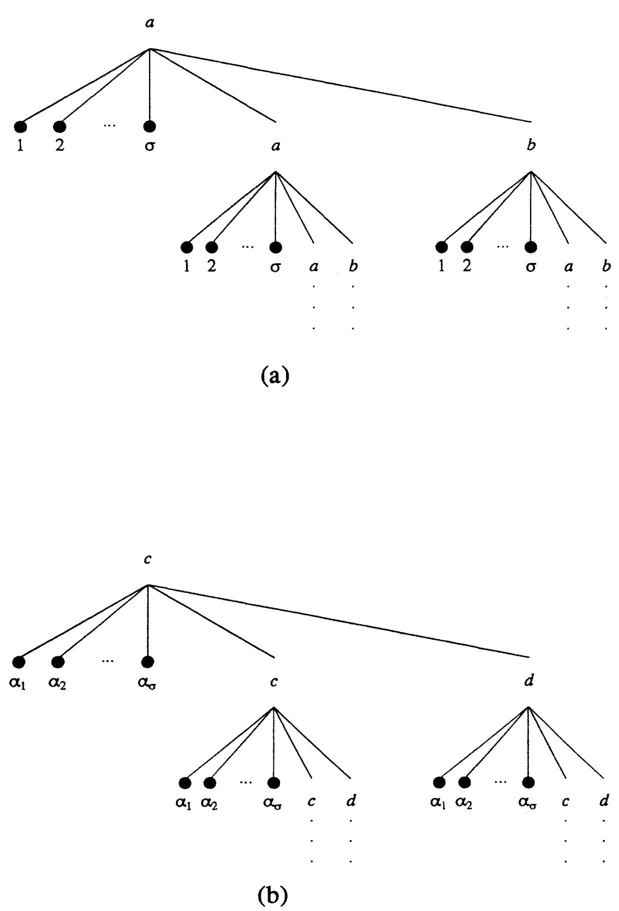


FIG. 3. Tree $\hat{T}_a$ of Eq. (A1), $\sigma=[(3k+1)/2]-\beta$ and α_i are defined in the Appendix.

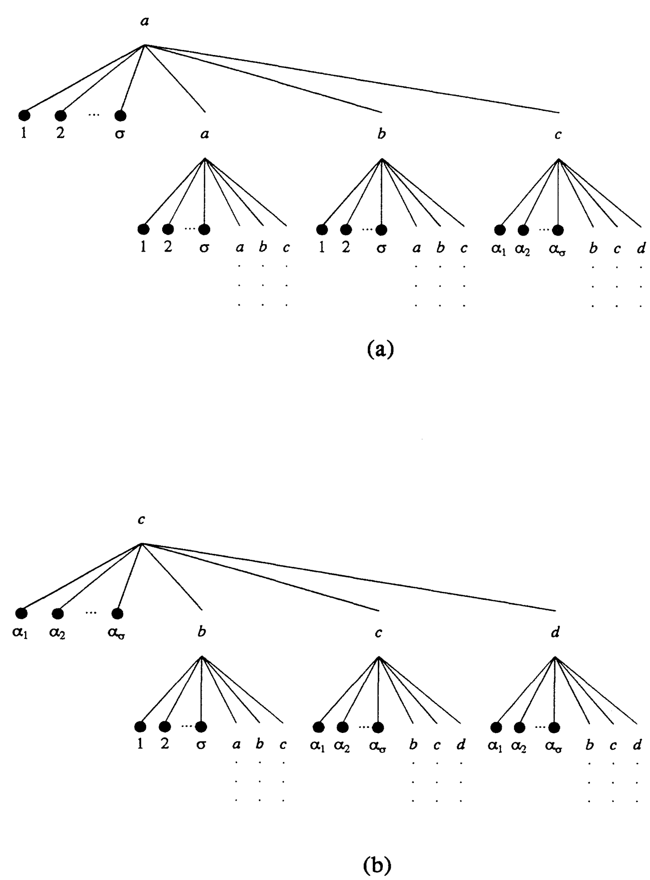
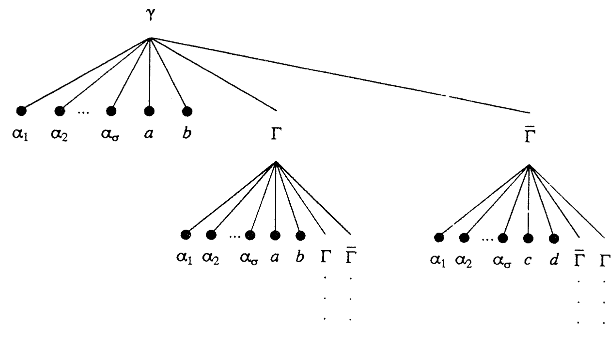
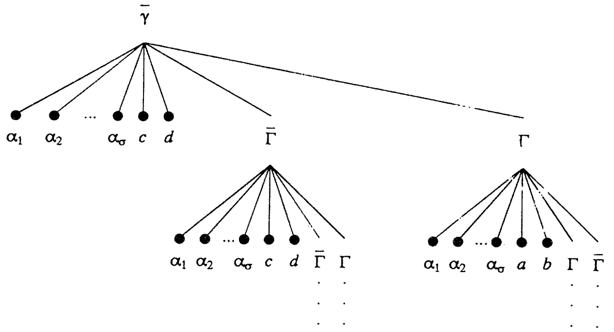


FIG. 4. Tree $\hat{T}_a$ of the case $\tilde{h}=[(3k+1)/2m]+3\lambda$ and α_i are defined in the Appendix.



(a)



(b)

FIG. 5. Tree $\hat{T}_\gamma$ and $\hat{T}_{\bar{\gamma}}$ of Eq. (A3). The notations are defined in the Appendix.

Fig. 3), only T_a and T_c are given in Fig. 3 in which $\alpha_i = K_+ + \beta + i$. T_b (T_d) can be obtained by replacing the first generation a (c) with b (d), respectively. For example, from Fig. 3, T_a contains the symbols $ai, aai, abi, \ldots,$ etc. The allowed prime cycles which are excluded by Φ are provided by Φ':

$$\Phi' = \{p | p \in T'_\alpha, \alpha = a,b,c,d\} \ , \tag{A2}$$

where T'_α is the set of all undotted elements in the tree $\hat{T}_\alpha$ given in Fig. 3, with $\sigma = (3k+1)/2$ and $\alpha_i = (k+3)/2, (k+5)/2, \ldots, 2k+1$.

(d) $\tilde{h} = (k+1+\beta/m) + 2\lambda$ ($\beta = 0,1,2,\ldots,K_-$). For this case, Φ and Φ' are the same as defined in (A1) and (A2). The tree $\hat{T}_\alpha$ is given in Fig. 3, with $\sigma = k+1+\beta$ and $\alpha_i = k+1-\beta, k+2-\beta, \ldots, 2k+1$.

(e) $\tilde{h} = [(3k+1)/2m] + 3\lambda$. The forbidden sequences are as follows: -$a\bar{\gamma}$-, -ad-, -$b\bar{\gamma}$-, -bd-, -$c\gamma$-, -ca-, -$d\gamma$-, -db-, where $\bar{\gamma} = (3k+3)/2, (3k+5)/2, \ldots, 2k+1$ and $\gamma = 1,2,\ldots,K_+$. In this case a new feature appears since $\hat{T}_\alpha$ starts to mix among $\{a,b\}$ and $\{c,d\}$. The sets Φ and Φ' are defined as before with T_α and T'_α given in Fig. 4, where $\sigma = (3k+1)/2$ and $\alpha_i = (3k+3)/2, (3k+5)/2, \ldots, 2k+1$.

(f) $\tilde{h} = [(3k+1+2\beta)/2m] + 4\lambda$ ($\beta = 0,1,\ldots,K_-$). The forbidden sequences are as follows: -$a\bar{\gamma}$, -$b\bar{\gamma}$-, -$c\gamma$-, -$d\gamma$-. Where $\bar{\gamma} = (3k+2\beta+3)/2$, $(2k+2\beta+5)/2, \ldots, 2k+1$ and $\gamma = 1,2,\ldots,K_+ - \beta$. The set Φ is given by

$$\Phi = \{i,s | s \in T_\gamma \text{ and } T_{\bar{\gamma}}\} \ . \tag{A3}$$

Here $i \in U - \{\gamma, \bar{\gamma}\}$; $U = \{1,2,\ldots,2k+1,a,b,c,d\}$ and $\{\gamma,\bar{\gamma}\} = \{1,2,\ldots,K_+ - \beta$; $(3k+2\beta+3)/2, (3k+2\beta+5)/2, \ldots, 2k+1\}$. $\hat{T}_\alpha$'s are given in Fig. 5, where $\alpha_i \in U' - \{\gamma,\bar{\gamma}\}; U' = \{1,2,\ldots,2k+1\}$. [$\gamma \in \Gamma = \{1,2,\ldots,K_+ - \beta\}$ and $\bar{\gamma} \in \bar{\Gamma} = \{(3k+2\beta+1)/2(3k+2\beta+3)/2, \ldots, 2k+1\}$. The line connecting γ and Γ is a shorthand notation for $K_+ - \beta$ lines; $\gamma\Gamma = \{\gamma 1, \gamma 2, \ldots \gamma(K_+ - \beta)\}$. Similarly, lines connecting two Γ's denote all combinations between all possible γ's. There are totally $K_+ - \beta$ different $\hat{T}'_\gamma$'s. Similarly, the same interpretation applies for $\hat{T}_{\bar{\gamma}}$.] Similarly Φ' is

$$\Phi' = \{p | p \in T'_\alpha\} \ , \tag{A4}$$

where T'_α is the set of all prime cycles generated by the undotted sequences of $\hat{T}_\alpha$.

[1] B. A. Huberman, J. P. Crutchfield, and N. H. Packard, Appl. Phys. Lett. **37**, 750 (1980); M. Cirillo and N. F. Pedersen, Phys. Lett. **90A**, 150 (1982); E. Ben-Jacob, I. Goldhirsch, Y. Imry, and S. Fishman, Phys. Rev. Lett. **49**, 1599 (1982).

[2] R. Artuso, Phys. Lett. A **160**, 528 (1991).

[3] R. Artuso, E. Aurell, and P. Cvitanovic, Nonlinearity **3**, 325 (1990) and references therein.

[4] H. C. Tseng, H. J. Chen, P. C. Li, W. Y. Lai, C. H. Chou, and H. W. Chen, Phys. Lett. A **195**, 74 (1994).

[5] S. Grossmann and H. Fujisaka, Phys. Rev. A **26**, 1779 (1982). A brief discussion on this work can be found in H. G. Schuster, *Deterministic Chaos* (Springer-Verlag, New York, 1984), Sec. 2.3, pp. 26–29.